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On the spin of the $B = 7$ skyrmion

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Abstract
We investigate the collective coordinate quantization of the icosahedrally symmetric $B = 7$ skyrmion, which is known to have a ground state with spin $\frac{7}{2}$ and isospin $\frac{1}{2}$. We find a particular quantum state maximally preserving the symmetries of the classical solution, and also present a novel relationship between the quantum state and the rational map approximation to the classical solution. We also investigate the allowed spin states if the icosahedral symmetry is partially broken. Skyrme field configurations with D_5 residual symmetry can be quantized with spin $\frac{3}{2}$, giving a realistic model for the ground states of the ^{7}Li/^{7}Be isospin doublet.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Skyrmions, first introduced in [1], are topological solitons in three spatial dimensions which are candidates for the description of nuclei, the baryon (or nucleon) number B being identified with the topological soliton number. Partly with the help of the rational map approximation [2], minimal energy skyrmion solutions have been found numerically for baryon numbers up to $B = 22$ and beyond, and their symmetries have been determined [3–5]. In these calculations a zero pion mass was usually assumed, and although some recent developments [6, 7] show that this approximation is not fully justified, for smaller B it remains a good model, and we will be using it in what follows. All minimal energy skyrmions with $B > 2$ have only discrete symmetries, i.e. are invariant under a discrete group of combined rotations and isorotations, and their baryon densities are localized around the edges of some polyhedra. After quantization of the collective rotational and isorotational degrees of freedom (and possibly some vibrational modes), these polyhedral density distributions are generally smoothed into a more spherical form which, one hopes, gives a good match with the experimental shapes of nuclei. A quite successful analysis was carried out in [8–13], showing that if one performs the collective coordinate quantization of skyrmions with baryon numbers $B = 2, 3, 4$, or 6, imposing the Finkelstein–Rubinstein (FR) constraints associated with the symmetries of the classical configurations [14], which encode the requirement that each $B = 1$ skyrmion be quantized as a fermion, then the minimal energy solutions will have the right spin/parity and isospin properties to model the deuteron, ^{3}H/^{3}He,^{4} He and ^{6}Li respectively.

The $B = 7$ solution [4], which is considered in this paper, is particularly symmetric, having icosahedral Y_h symmetry, with the baryon density being localized around the edges of a dodecahedron. The collective coordinate quantization has been considered in detail by Irwin [12] and Krusch [15], with the help of the rational map approximation. However, the lowest allowed spin state obtained with this approach is $J = \frac{7}{2}$ (with the isospin being $I = \frac{1}{2}$), which gives rise to a disagreement with real nuclei. Experimentally, $J = \frac{7}{2}$ appears as the second excited state of the ^{7}Li/^{7}Be doublet, with an excitation energy of 4.6 MeV (relatively low for such a high spin), whereas the ground state has spin $J = \frac{3}{2}$ and the nearby first excited state has spin $J = \frac{1}{2}$. This suggests that the $B = 7$ dodecahedral skyrmion is too symmetric to describe the physical ground state, and the icosahedral group should be partially broken to allow for states with smaller spin. A deformed skyrmion will have a larger classical potential energy than the undeformed one, but could be energetically preferred because it would be quantized with a lower spin and hence have lower kinetic energy. The kinetic energy associated with the $J = \frac{7}{2}$ state has never been calculated, but is of order 7–12 MeV. A $J = \frac{3}{2}$ state has kinetic energy of order 2–5 MeV, so the potential energy of the deformed configuration might be up to 15 MeV, still allowing for a lower total energy.

This paper is organized as follows. In section 2 we review the basic concepts of the Skyrme model and the rational map ansatz, which we use in the later sections. Then in section 3 we present the rational map for the $B = 7$ skyrmion, in various orientations. In section 4 we review the quantization of the $B = 7$ skyrmion, and, following [13], show using a slightly modified approach to the quantized spin $\frac{7}{2}$ state of the skyrmion that the dodecahedral density can be substantially preserved even in the quantum case. We also show that the rational map itself contains useful information about the quantum state. In section 5 we examine different choices for breaking icosahedral symmetry, find new ground states with various spins, and find the corresponding collective coordinate wavefunctions. In particular, we show that if one breaks the D_3 subgroup of Y_h, while still preserving D_5 symmetry, the ground state will be the observed $\left(J = \frac{3}{2}, I = \frac{1}{2}\right)$ state. Alternatively, if the D_5 subgroup is broken, both $\left(J = \frac{3}{2}, I = \frac{1}{2}\right)$ and $\left(J = \frac{1}{2}, I = \frac{1}{2}\right)$ are allowed.

2. Skyrmions and the rational map ansatz

In dimensionless units the Skyrme model with zero pion mass has Lagrangian

$$L = \int \left\{ \frac{1}{2} \mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{1}{16} \mathrm{Tr}([\partial_\mu U U^\dagger, \partial_\nu U U^\dagger][\partial^\mu U U^\dagger, \partial^\nu U U^\dagger]) \right\} \mathrm{d}^3x, \tag{1}$$

where $U(t, \mathbf{x})$ is an $SU(2)$-valued scalar field, which can be expressed nonlinearly in terms of the isospin triplet of massless pion fields, and satisfies the boundary condition $U(\mathbf{x}) \to 1_2$ as $\mathbf{x} \to \infty$. This boundary condition implies that U can be regarded as a map $U : S^3 \to S^3$, where the domain S^3 is identified with $\mathbb{R}^3 \bigcup \{\infty\}$, and the target S^3 is the manifold of $SU(2)$. The topological degree of the map U has the explicit representation

$$B = -\frac{1}{24\pi^2} \int \varepsilon_{ijk} \mathrm{Tr}(\partial_i U U^\dagger \partial_j U U^\dagger \partial_k U U^\dagger) \, \mathrm{d}^3x. \tag{2}$$

The conservation of the topological invariant B makes it possible to identify the solutions of the Skyrme field equation with classical nuclei, with B standing for the baryon number. The lowest energy static solutions of the model, for each B, are called skyrmions.

Rational maps, i.e. holomorphic maps from $S^2 \to S^2$, prove to give good approximations to skyrmion solutions, especially those with low baryon number. One identifies the domain S^2 of the rational map with a sphere in $\mathbb{R}^3$ centred at the origin, of indeterminate radius, and the target S^2 with the unit sphere in the Lie algebra of $SU(2)$. $\mathbb{R}^3$ can be given coordinates (r, z), where r denotes the radius and the complex variable z denotes $\tan\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\phi}$ (the stereographic coordinate, with θ and ϕ the usual polar angles). The rational map $R(z)$ is a ratio of polynomials in z. Its value R, at any point, corresponds (by stereographic projection) to the Cartesian unit vector

$$\mathbf{n}_R = \frac{1}{1+|R|^2}(R+\bar{R}, \mathrm{i}(\bar{R}-R), 1-|R|^2). \tag{3}$$

The rational map ansatz for the Skyrme field, depending on $R(z)$ and a radial profile function $f(r)$, is

$$U(r, z) = \exp(\mathrm{i} f(r)\mathbf{n}_{R(z)} \cdot \boldsymbol{\tau}), \tag{4}$$

where $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ is the triplet of Pauli matrices, and $f(r)$ satisfies $f(0) = \pi$, $f(\infty) = 0$.

The baryon number for the ansatz (4) is given by

$$B = -\int \frac{f'}{2\pi^2}\left(\frac{\sin f}{r}\right)^2 \left(\frac{1+|z|^2}{1+|R|^2}\left|\frac{\mathrm{d}R}{\mathrm{d}z}\right|\right)^2 \frac{2\mathrm{i}\,\mathrm{d}z\,\mathrm{d}\bar{z}}{(1+|z|^2)^2} r^2\,\mathrm{d}r, \tag{5}$$

where $2\mathrm{i}\,\mathrm{d}z\,\mathrm{d}\bar{z}/(1+|z|^2)^2$ is equivalent to the usual 2-sphere area element $\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$. The angular part of the integrand,

$$\left(\frac{1+|z|^2}{1+|R|^2}\left|\frac{\mathrm{d}R}{\mathrm{d}z}\right|\right)^2, \tag{6}$$

multiplied by the area form $2\mathrm{i}\,\mathrm{d}z\,\mathrm{d}\bar{z}/(1+|z|^2)^2$, is precisely the pull-back of the area form $2\mathrm{i}\,\mathrm{d}R\,\mathrm{d}\bar{R}/(1+|R|^2)^2$ on the target sphere of the rational map, so its integral is 4π times the degree N of the map. Therefore, with our choice for the boundary conditions of f, equation (5) simplifies to

$$B = \frac{-2N}{\pi}\int_0^\infty f'\sin^2 f\,\mathrm{d}r = N, \tag{7}$$

in other words, a rational map of degree B gives a skyrmion of baryon number B.

An $SU(2)$ Möbius transformation of z corresponds to a rotation in physical space; an $SU(2)$ Möbius transformation of R (i.e. on the target S^2) corresponds to an isospin rotation. Both are symmetries of the Skyrme model, and preserve the baryon number and energy. It is the rotational and isorotational collective coordinates that we need to quantize. (There is also translational symmetry in the Skyrme model, but its quantization is standard, leading to momentum eigenstates.) Because of the correspondence between the Möbius transformation and an ordinary rotation, these coordinates may be taken to be the Euler angles (α, β, γ). This choice will prove useful later.

An attractive feature of the rational map ansatz is that it leads to a simple classical energy expression which can be separately minimized with respect to the parameters of the rational map $R(z)$ and the profile function $f(r)$ to obtain close approximations to the numerical, exact skyrmion solutions, and almost always with the correct symmetries. For some small values of B, including $B = 7$, there is a unique rational map of the desired degree with the correct symmetry, which also minimizes the angular part of the energy.

After quantizing the Skyrme field, much of the interesting information, including the symmetries of the quantum baryon density, is encoded in its angular part which only depends on the rational map; therefore the profile function f will not be of much interest for our purposes.

3. Icosahedrally symmetric $B = 7$ skyrmion

The minimal energy $B = 7$ skyrmion has a dodecahedral shape, with holes in the baryon density at the centres of the faces [4]. The symmetry group is the icosahedral group Y_h, whose rotational subgroup is generated by a $2\pi/5$ rotation about a face, and a $2\pi/3$ rotation about a vertex attached to that face. Their product generates a π rotation about the midpoint of an edge attached to the face.

The skyrmion can be well approximated by the rational map ansatz, and there is an essentially unique Y_h-symmetric map of degree 7, depending only on the choice of orientation in space and isospace. One can orient the skyrmion, so that one or other of the above symmetry generators is manifest, and the map then takes the following concise forms.

(1) C_5 symmetry manifest [2]:

$$R(z) = \frac{7z^5 + 1}{z^2(z^5 - 7)}. \tag{8}$$

(2) C_2 symmetry manifest [2]:

$$R(z) = \frac{bz^6 - 7z^4 - bz^2 - 1}{z(z^6 + bz^4 + 7z^2 - b)}, \qquad b = 7/\sqrt{5}. \tag{9}$$

(3) For manifest C_3 symmetry the corresponding rational map has not previously been found, and we obtain it here. Recall that the Wronskian of a rational map

$$R(z) = \frac{P(z)}{Q(z)} \tag{10}$$

is defined as

$$W(z) = P'(z)Q(z) - P(z)Q'(z). \tag{11}$$

We can rotate the Wronskian defined for one of the previous orientations, and use this to find the required map.

Consider the rational map (8), whose Wronskian is a multiple of

$$W(z) = z^{11} + 11z^6 - z. \tag{12}$$

The Wronskian of a general, degree 7 rational map is a 12th-order polynomial; therefore in our case the zeros of (12) are defined by $\{z : W(z) = 0\} \bigcup \{z = \infty\}$, and they are situated at the face centres of the dodecahedron, the points of zero baryon density. One of these face centres is at $z = 0$. We now seek a rotation which moves one of the vertices of the dodecahedron to $z = 0$. From [16] we know that (in a certain orientation) the Cartesian unit vectors pointing to a vertex and nearest face center are

$$\mathbf{n}_v = \frac{1}{\sqrt{3}}(0, \tau^{-1}, \tau) \qquad \text{and} \qquad \mathbf{n}_f = \frac{1}{\sqrt{1+\tau^2}}(0, \tau, 1), \tag{13}$$

respectively, with $\tau = (1 + \sqrt{5})/2$ the golden ratio. Therefore, the rotation angle is given by

$$\cos\lambda = \mathbf{n}_v \cdot \mathbf{n}_f = \frac{\tau^2}{\sqrt{3(1+\tau^2)}}. \tag{14}$$

The corresponding Möbius transformation for a rotation by the above angle, preserving the real z-axis, is

$$z \to \tilde{z} = \frac{z\cos\frac{\lambda}{2} - \sin\frac{\lambda}{2}}{z\sin\frac{\lambda}{2} + \cos\frac{\lambda}{2}}. \tag{15}$$

Acting with (15) on the Wronskian (12) we again get an 11th-order polynomial in the numerator. We must multiply this by the factor $z + \cot\frac{\lambda}{2}$ corresponding to the zero that has rotated from $z = \infty$. The result is the 12th-order polynomial with manifest C_3 symmetry:

$$W(z) = z^{12} + 11\sqrt{5}z^9 - 33z^6 - 11\sqrt{5}z^3 + 1. \tag{16}$$

The corresponding rational map $P(z)/Q(z)$, where one of $P(z)$ and $Q(z)$ is of degree 7 and the other of degree 7 or less, can be worked out by solving (11). This is a system of 12 equations in 15 variables, so there is some ambiguity, but it disappears once we require manifest C_3 symmetry; this leads to the unique rational map

$$R(z) = \frac{7z^6 + 7\sqrt{5}z^3 + 2}{z(2z^6 - 7\sqrt{5}z^3 + 7)}. \tag{17}$$

4. Quantization

4.1. FR constraints

We quantize the $B = 7$ skyrmion as a rigid body free to rotate in space and isospace, and assume here that it has its undistorted dodecahedral shape. The symmetries of the skyrmion impose restrictions on the allowed quantum states via the FR constraints [14]. For each symmetry element given by a rotation by α in space accompanied by an isospin rotation by β in the target space, we have to impose the following condition on the wavefunction Ψ:

$$\exp(\mathrm{i}\alpha\mathbf{n}\cdot\mathbf{L})\exp(\mathrm{i}\beta\mathbf{N}\cdot\mathbf{K})\Psi = \chi_{\mathrm{FR}}\Psi, \tag{18}$$

with $\mathbf{n}$ and $\mathbf{N}$ being the directions of the rotation axes in space and isospace, respectively, and $\mathbf{L}$ and $\mathbf{K}$ the spin and isospin operators with respect to body-fixed axes. As usual for a rigid body, the spin and isospin operators with respect to axes fixed in space, $\mathbf{J}$ and $\mathbf{I}$, are distinct, but the Casimirs are the same: $\mathbf{J}^2 = \mathbf{L}^2$ and $\mathbf{I}^2 = \mathbf{K}^2$.

For a skyrmion whose symmetry is captured by the rational map ansatz, the factor χ_{FR}, which is ± 1, can be neatly evaluated using a formula due to Krusch [15]:

$$\chi_{\mathrm{FR}} = (-1)^{\mathcal{N}} \qquad \text{where} \quad \mathcal{N} = B(B\alpha - \beta)/2\pi. \tag{19}$$

The rational map itself determines the correct sign choice for the directions of the rotation axes and hence the signs of the angles of rotation. Let $z_{\pm\mathbf{n}}$ denote the stereographic coordinates corresponding to $\pm\mathbf{n}$, and similarly $R_{\pm\mathbf{N}}$ those corresponding to $\pm\mathbf{N}$. The symmetry of the rational map $R(z)$ implies that $R(z_{-\mathbf{n}})$ is one of $R_{\pm\mathbf{N}}$. Having chosen $\mathbf{n}$, one should choose $\mathbf{N}$ so that $R(z_{-\mathbf{n}}) = R_{\mathbf{N}}$ [15]. There is another ambiguity for odd baryon numbers: namely one might add 2π to the rotation or isorotation angles. This will not affect the overall result, though, as the additional minus sign associated with a 2π rotation or isorotation will be compensated by the extra minus sign coming from χ_{FR} according to the formula (19).

Looking carefully at the rational maps (8), (9) and (17), with icosahedral symmetry in various orientations, one sees that a $2\pi/5$, π or $2\pi/3$ rotation about the x_3-axis, accompanied by an isorotation by $4\pi/5$, π or $2\pi/3$, respectively, about the third isospin axis (both axes pointing up), leaves the rational map unchanged. Therefore,

$$\begin{aligned}\chi_{\rm FR} &= -1, \quad \text{for the } C_5 \text{ generator},\\ \chi_{\rm FR} &= -1, \quad \text{for the } C_2 \text{ generator},\\ \chi_{\rm FR} &= +1, \quad \text{for the } C_3 \text{ generator}.\end{aligned} \tag{20}$$

Note that the product of the first two generators gives the third, and the FR sign factors give a representation of this. More generally, imposing the FR constraints for these generators extends consistently to the whole Y_h symmetry group. In [12] it was shown that these constraints force the ground state to have isospin $I = \frac{1}{2}$, and spin $J = \frac{7}{2}$. The wavefunction, corresponding to the manifestly C_5-symmetric orientation specified by (8), is

$$|\Psi\rangle = \left\{\sqrt{\tfrac{7}{10}}\left|\tfrac{7}{2}, -\tfrac{3}{2}\right\rangle - \sqrt{\tfrac{3}{10}}\left|\tfrac{7}{2}, \tfrac{7}{2}\right\rangle\right\} \otimes \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle + \left\{\sqrt{\tfrac{7}{10}}\left|\tfrac{7}{2}, \tfrac{3}{2}\right\rangle + \sqrt{\tfrac{3}{10}}\left|\tfrac{7}{2}, -\tfrac{7}{2}\right\rangle\right\} \otimes \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle. \tag{21}$$

Here, the terms in braces are the spin parts of the wavefunction, and the second entry, after the total spin $\frac{7}{2}$, is the spin projection onto the third body axis. (Note that because of the C_5 symmetry, these values differ by 5.) These spin parts are tensored with the isospin parts, where the second entry is the (apparently unobservable) projection on to the third 'body' isoaxis. Here, we do not specify the projection of the total spin on to the third space axis, as this is arbitrary. The projection of isospin on to the third 'space' isoaxis can be either $\frac{1}{2}$ or $-\frac{1}{2}$, giving a ^{7}Be or ^{7}Li state, and is also not specified.

More explicitly, the wavefunction (21) can be expressed in terms of Wigner functions of the rotational and isorotational Euler angles. Ψ is then the amplitude to find the skyrmion in the orientation with those Euler angles relative to the standard orientation of the rational map (8).

Since we are also interested in having the C_3 symmetry manifest, we have calculated the wavefunction when the standard orientation is that specified by (17). This is

$$\begin{aligned}|\Psi\rangle = {}&\left\{-\tfrac{\sqrt{2}}{3}\left|\tfrac{7}{2}, \tfrac{7}{2}\right\rangle + \sqrt{\tfrac{7}{18}}\left|\tfrac{7}{2}, \tfrac{1}{2}\right\rangle - \sqrt{\tfrac{7}{18}}\left|\tfrac{7}{2}, -\tfrac{5}{2}\right\rangle\right\} \otimes \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle\\ &+ \left\{\sqrt{\tfrac{7}{18}}\left|\tfrac{7}{2}, \tfrac{5}{2}\right\rangle + \sqrt{\tfrac{7}{18}}\left|\tfrac{7}{2}, -\tfrac{1}{2}\right\rangle + \tfrac{\sqrt{2}}{3}\left|\tfrac{7}{2}, -\tfrac{7}{2}\right\rangle\right\} \otimes \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle.\end{aligned} \tag{22}$$

We have found a novel way to verify the structures of the wavefunctions (21) or (22). The coefficients are essentially the same as occur in the rational map itself. Observe that for $B = 7$ the rational map has a numerator which is, in general, a degree 7 polynomial in z, with eight coefficients. The denominator is similar. Under a Möbius transformation of z, corresponding to a rotation, the numerator and denominator each change, with the coefficients transforming according to the eight-dimensional representation of $SU(2)$. This is the spin $\frac{7}{2}$ representation. Similarly, under an isorotational Möbius transformation, the numerator and denominator are mixed by the fundamental isospin $\frac{1}{2}$ representation. Therefore, under rotations and isorotations, the quantum wavefunction with spin $\frac{7}{2}$ and isospin $\frac{1}{2}$ transforms just like the rational map. Furthermore, we require the quantum wavefunction to have, with respect to body-fixed axes, the symmetries of the rational map, i.e. the wavefunction is unchanged under each symmetry operation, up to an FR sign factor. Again, the rational map itself has precisely these properties. So, to get the wavefunction, we just take the coefficients of the rational map, and identify them with the coefficients of the wavefunction. The one further step is to correctly identify and normalize the basis elements.

For this last step, we note that a general degree 7 rational map may be written in the form

$$R(z) = \frac{\sum_{s=-\frac{7}{2}}^{s=\frac{7}{2}} P_s z^s}{\sum_{s=-\frac{7}{2}}^{s=\frac{7}{2}} Q_s z^s}, \tag{23}$$

where s takes half odd-integer values. Taking here $z = \tan\frac{\beta}{2}\,e^{i\gamma}$, we can identify each monomial in z in (23) with a Wigner function $D^{\frac{7}{2}}_{-s,\frac{7}{2}}$ depending on the Euler angles α, β, γ, in which the projection of the spin onto the third space axis is maximal. More precisely, [17]

$$D^{\frac{7}{2}}_{-s,\frac{7}{2}}(\alpha, \beta, -\gamma) = e^{i\frac{7}{2}\alpha}\left(\frac{\sin\beta}{2}\right)^{7/2}(-1)^{\frac{7}{2}+s}\left(\frac{7!}{\left(\frac{7}{2}-s\right)!\left(\frac{7}{2}+s\right)!}\right)^{1/2} z^{s}. \quad (24)$$

So if we multiply the numerator and denominator of the rational map (23) by the common factor $e^{i\frac{7}{2}\alpha}\left(\frac{\sin\beta}{2}\right)^{7/2}$, then it becomes a ratio of sums of Wigner functions. The numerator and denominator become the (body) isospin-down and isospin-up parts of the wavefunction. For example, the powers z^7 and z^2 in the denominator of the rational map (8) correlate with the spin projection terms $-\frac{7}{2}$ and $\frac{3}{2}$ in the wavefunction (21). Also the ratio of coefficients in the rational map, $1 : -7$, converts to the ratio of coefficients in the wavefunction $1 : \sqrt{7/3}$ because of the normalization factors of the Wigner functions $D^{\frac{7}{2}}_{-\frac{7}{2},\frac{7}{2}}$ and $D^{\frac{7}{2}}_{\frac{3}{2},\frac{7}{2}}$.

This new way to obtain wavefunctions directly from the rational map gives an unexpected significance to the rational map approximation to skyrmions. It means that the rational map ansatz is not only a tool to find approximate classical solutions, but may also encode exact information about quantum states. Unfortunately, this happens rather rarely. It only works for odd baryon numbers, and where the isospin is $\frac{1}{2}$ and the spin is $\frac{B}{2}$ (the dimensions of these $SU(2)$ representations being 2 and $B + 1$). The only other frequently occurring example is the ground state of the $B = 1$ skyrmion, with spin and isospin $\frac{1}{2}$.

4.2. Classical and quantum baryon density

In [13] we have noted that there is some choice for the quantum baryon density of a skyrmion. For example, the shape of the quantum state will depend on the spin projection on to the third spatial axis, m, which is usually considered arbitrary. Here we follow the same logic as in [13] and construct a quantum state Ψ of the $B = 7$ icosahedrally symmetric skyrmion which retains as much as possible of the spatial symmetry of the classical solution. We do this by taking a suitable linear combination of states with different m. The quantum baryon density is then found by averaging the classical baryon density over the collective coordinates weighted with $|\Psi|^2$. In what follows, we will restrict our calculations to the states (21), based on the rational map (8). The states (21) can be rewritten in terms of Wigner functions,

$$|\Psi_m\rangle = |\psi_m\rangle \otimes \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle + |\chi_m\rangle \otimes \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle, \quad (25)$$

where

$$\begin{aligned} |\psi_m\rangle &= \sqrt{\tfrac{7}{10}}D^{\frac{7}{2}}_{-\frac{3}{2},m}(\alpha, \beta, \gamma) - \sqrt{\tfrac{3}{10}}D^{\frac{7}{2}}_{\frac{7}{2},m}(\alpha, \beta, \gamma), \\ |\chi_m\rangle &= \sqrt{\tfrac{7}{10}}D^{\frac{7}{2}}_{\frac{3}{2},m}(\alpha, \beta, \gamma) + \sqrt{\tfrac{3}{10}}D^{\frac{7}{2}}_{-\frac{7}{2},m}(\alpha, \beta, \gamma). \end{aligned} \quad (26)$$

The spatial spin projection m is now explicit, and α, β, γ are the spatial Euler angles. From the orthogonality of different spin and isospin states it follows that the required wavefunction, i.e. the one which is icosahedrally symmetric with respect to both body-fixed and space-fixed axes, is the combination

$$|\Psi\rangle = \left\{\sqrt{\tfrac{7}{10}}\left|\psi_{-\frac{3}{2}}\right\rangle - \sqrt{\tfrac{3}{10}}\left|\psi_{\frac{7}{2}}\right\rangle\right\} \otimes \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle + \left\{\sqrt{\tfrac{7}{10}}\left|\chi_{\frac{3}{2}}\right\rangle + \sqrt{\tfrac{3}{10}}\left|\chi_{-\frac{7}{2}}\right\rangle\right\} \otimes \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle. \quad (27)$$

Note that the combination of m-values that is needed is just the same combination as occurs in (21).

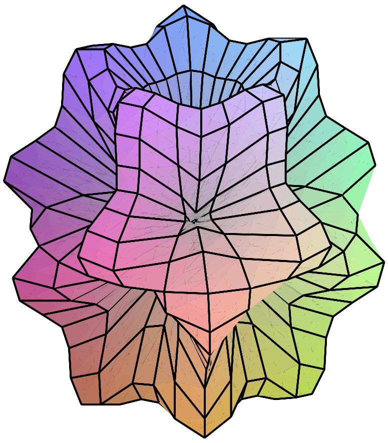

Figure 1. Classical baryon density of $B = 7$ skyrmion, truncated to first four terms in expansion (33).

The quantum baryon density is defined as

$$\rho_\Psi(\mathbf{x}) = \int \mathcal{B}(D(A)^{-1}\mathbf{x})|\Psi(A)|^2 \sin\beta \, \mathrm{d}\alpha \, \mathrm{d}\beta \, \mathrm{d}\gamma, \tag{28}$$

where $\Psi(A)$ is the normalized wavefunction. Here A stands for the $SU(2)$ matrix parametrized by Euler angles α, β, γ, and $D(A)$ for the $SO(3)$ matrix associated with A via

$$D(A)_{ab} = \tfrac{1}{2}\operatorname{Tr}(\tau_a A \tau_b A^\dagger), \tag{29}$$

and $\mathcal{B}(\mathbf{x})$ is the classical baryon density of the $B = 7$ skyrmion. The important part is the angular dependence of $\mathcal{B}(\mathbf{x})$, obtained by evaluating (6) for the rational map (8):

$$\mathcal{B} = \frac{196|z|^2(1+|z|^2)^2(z^{10}+11z^5-1)(\bar{z}^{10}+11\bar{z}^5-1)}{(|z|^{14}+49|z|^{10}+49|z|^4+1-7(|z|^4-1)(z^5+\bar{z}^5))^2}. \tag{30}$$

Expressed in terms of polar angles,

$$\mathcal{B} = 196\tan^2\frac{\theta}{2}\left(1+\tan^2\frac{\theta}{2}\right)^2 \times \frac{\tan^{20}\frac{\theta}{2} + 22\tan^{15}\frac{\theta}{2}\cos 5\phi - 2\tan^{10}\frac{\theta}{2}\cos 10\phi + 121\tan^{10}\frac{\theta}{2} - 22\tan^5\frac{\theta}{2}\cos 5\phi + 1}{\left(\tan^{14}\frac{\theta}{2} + 49\tan^{10}\frac{\theta}{2} - 14\tan^9\frac{\theta}{2}\cos 5\phi + 14\tan^5\frac{\theta}{2}\cos 5\phi + 49\tan^4\frac{\theta}{2} + 1\right)^2}. \tag{31}$$

It is convenient to expand $\mathcal{B}$ in terms of spherical harmonics $Y_{lm}(\theta, \phi)$,

$$\mathcal{B} = \sum_{l,m} c_{lm} Y_{lm}(\theta, \phi), \tag{32}$$

where because of the icosahedral symmetry the lowest values of l are 0, 6 and 10, and the values of m are multiples of 5. The infinite series is dominated by the first four terms,

$$\mathcal{B} = c_{00}Y_{00} + c_{6-5}Y_{6-5} + c_{60}Y_{60} + c_{65}Y_{65} + \cdots, \tag{33}$$

and all higher terms contribute less than a 10% correction. Because the map (8) has degree 7, the integral of $\mathcal{B}$ over the sphere is 28π, so $c_{00} = 14\sqrt{\pi}$, and we find numerically $c_{65} = -c_{6-5} \simeq -6.38$, $c_{60} \simeq 7.97$ (see figure 1). The ratio c_{60}/c_{65} may also be found

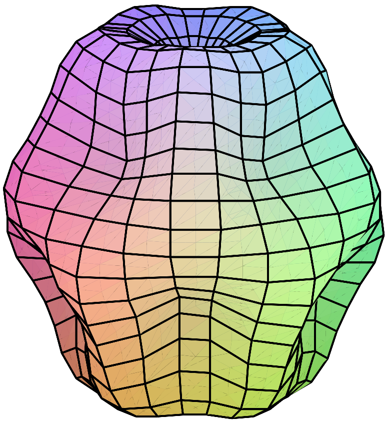

Figure 2. Quantum baryon density of $B = 7$ skyrmion in spin $\frac{7}{2}$ state.

analytically from the fact that

$$Z = c_{6-5}Y_{6-5} + c_{60}Y_{60} + c_{65}Y_{65} \tag{34}$$

is Y_h-symmetric and should have equal values at all the Wronskian points (i.e. the dodecahedral face centres). In the orientation we are considering, the zeros of the Wronskian are $z = 0, z = \infty$, and the solutions of $z^{10} + 11z^5 - 1 = 0$. Solving this equation for z^5, and then explicitly calculating the fifth root, we find that the real, non-trivial Wronskian points are $z = (-1 \pm \sqrt{5})/2$. At $z = (-1 + \sqrt{5})/2$, $\cos\theta = 1/\sqrt{5}$ and $\phi = 0$, and we calculate that $Y_{65} \cong 0.428$, $Y_{60} \cong -0.334$. On the other hand, at $z = 0$, $Y_{65} = 0$ and $Y_{60} \cong -1.017$. Hence, from (34) we find that $c_{60}/c_{65} \cong -1.25$, which agrees with the numerical determination.

Using the transformation properties of spherical harmonics under rotations,

$$Y_{lm}(\tilde{\theta}, \tilde{\phi}) = \sum_k D^l_{mk}(A)^* Y_{lk}(\theta, \phi), \qquad \text{(no sum on } l\text{)}, \tag{35}$$

we can determine the rotated baryon density in the integrand of (28), and then using the integrals of the Wigner functions

$$\begin{aligned} &\int D^j_{ab}(A) D^{j'}_{cd}(A)^* \sin\beta \, \mathrm{d}\alpha \, \mathrm{d}\beta \, \mathrm{d}\gamma = \frac{8\pi^2}{2j+1} \delta^{jj'} \delta_{ac} \delta_{bd}, \\ &\int D^j_{ab}(A) D^{j'}_{cd}(A) D^{j''}_{ef}(A) \sin\beta \, \mathrm{d}\alpha \, \mathrm{d}\beta \, \mathrm{d}\gamma = 8\pi^2 \begin{pmatrix} j & j' & j'' \\ a & c & e \end{pmatrix} \begin{pmatrix} j & j' & j'' \\ b & d & f \end{pmatrix}, \end{aligned} \tag{36}$$

we find that the angular dependence of ρ_Ψ, the baryon density in the quantum state (27), is

$$\rho_\Psi = c_{00}Y_{00} + 0.23\,(c_{6-5}Y_{6-5} + c_{60}Y_{60} + c_{65}Y_{65}) \,. \tag{37}$$

This is a closed expression, since the higher order terms in the sum (33) all average out to zero, and it resembles the classical density, although more dominated by the first term, as can be seen from figure 2.

5. Breaking the icosahedral symmetry

The state with spin $\frac{7}{2}$ considered so far is not the observed ground state of ^{7}Be or ^{7}Li, the nuclei which should be described by the $B = 7$ skyrmion. We know from experiment that the ground states form an isospin doublet with spin $\frac{3}{2}$, and there is a nearby excited state with spin $\frac{1}{2}$. It is encouraging that at energy only 4.6 MeV above the ground state there is a spin $\frac{7}{2}$ state, as this is not a high energy for such a large spin. Nevertheless, we still have the problem of understanding the lower energy, lower spin states. One way to proceed is to break some of the symmetries, thus allowing more spin states. We seek a deformed $B = 7$ skyrmion, with a higher classical potential energy than the dodecahedral solution, but where the kinetic energy associated with the spin is lower.

In what follows, we shall just investigate the symmetries and allowed spins, without seriously searching for the state of lowest total energy. We will only be considering small perturbations of the dodecahedral skyrmion in order still to be able to apply the rational map ansatz, although ultimately one should consider more general deformations. Following Houghton and Magee [18], who investigated the deformation of the $B = 1$ skyrmion when it is in a state with spin $\frac{1}{2}$, we consider only the collective coordinate quantization of the deformed skyrmion, and we assume that the FR constraint associated with any unbroken symmetry is as before. We do not treat the deformation itself dynamically, since to do so consistently would require consideration of very many other shape deformation modes.

(1) *D_3 symmetry preserved.* D_3 is generated by C_2 and C_3 symmetries, the axes of which are orthogonal. We assume that the skyrmion is in the orientation of the rational map (17) with C_3 symmetry about the x_3-axis and C_2 symmetry about the x_2-axis. The C_5 symmetry about any face centre of the dodecahedron is broken, if, for example, the coefficient $7\sqrt{5}$ in the numerator and denominator of (17) is increased or decreased.

The FR constraints take the form (after checking the direction and sign of $\mathbf{N}$ in each case)

$$e^{\frac{2\pi i}{3}(L_3+K_3)}|\Psi\rangle = |\Psi\rangle, \qquad e^{\pi i(L_2+K_2)}|\Psi\rangle = |\Psi\rangle, \tag{38}$$

where L_i and K_i are generators of spin and isospin rotations respectively. The lowest allowed state is $J = \frac{1}{2}$, $I = \frac{1}{2}$, with wavefunction

$$|\Psi\rangle = \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle \otimes \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle - \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle \otimes \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle. \tag{39}$$

Spin $\frac{3}{2}$ is also allowed leading to

$$|\Psi\rangle = \left|\tfrac{3}{2}, \tfrac{1}{2}\right\rangle \otimes \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle + \left|\tfrac{3}{2}, -\tfrac{1}{2}\right\rangle \otimes \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle. \tag{40}$$

(2) *D_5 symmetry preserved.* This is generated by C_2 and C_5 symmetries about orthogonal axes. Here, we assume that the skyrmion is in the orientation of the rational map (8) with C_5 symmetry about the x_3-axis and C_2 symmetry about the x_2-axis. The C_3 symmetry is broken if the coefficient 7 in the numerator and denominator of (8) is varied. The FR constraints are

$$e^{\frac{2\pi i}{5}(L_3+2K_3)}|\Psi\rangle = -|\Psi\rangle, \qquad e^{\pi i(L_2+K_2)}|\Psi\rangle = |\Psi\rangle. \tag{41}$$

This is the most interesting case, as the lowest allowed state with isospin $\frac{1}{2}$ has spin $\frac{3}{2}$. The corresponding wavefunction is

$$|\Psi\rangle = \left|\tfrac{3}{2}, \tfrac{3}{2}\right\rangle \otimes \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle + \left|\tfrac{3}{2}, -\tfrac{3}{2}\right\rangle \otimes \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle. \tag{42}$$

(3) The degree 7 rational map

$$R(z) = \frac{bz^6 - 7z^4 - bz^2 - 1}{z(z^6 + bz^4 + 7z^2 - b)} \tag{43}$$

has icosahedral symmetry when $b = \pm 7/\sqrt{5}$, but for general real values of b there is only tetrahedral T_h symmetry. This symmetry is generated by a π rotation about the x_3-axis and a $2\pi/3$ rotation permuting the Cartesian axes. Therefore, we obtain the following FR constraints:

$$\begin{aligned} &\mathrm{e}^{\frac{2\pi\mathrm{i}}{3\sqrt{3}}(L_1+L_2+L_3)}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{3\sqrt{3}}(K_1+K_2+K_3)}|\Psi\rangle = |\Psi\rangle,\\ &\mathrm{e}^{\pi\mathrm{i}(L_3+K_3)}|\Psi\rangle = -|\Psi\rangle. \end{aligned} \tag{44}$$

This time the lowest allowed state is $J = \frac{1}{2}$, $I = \frac{1}{2}$, and the corresponding wavefunction is

$$|\Psi\rangle = \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle \otimes \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle + i\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle \otimes \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle. \tag{45}$$

There is no $J = \frac{3}{2}$, $I = \frac{1}{2}$ state.

A further special case is when $b = 0$, since the rational map

$$R(z) = -\frac{7z^4 + 1}{z^3(z^4 + 7)} \tag{46}$$

has cubic symmetry. The cubic group is generated by a $2\pi/3$ rotation cyclically permuting the Cartesian axes and a $\pi/2$ rotation about the x_3-axis. Applying these to (46), one finds the accompanying isospin rotations. The corresponding FR constraints are

$$\begin{aligned} &\mathrm{e}^{\frac{2\pi\mathrm{i}}{3\sqrt{3}}(L_1+L_2+L_3)}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{3\sqrt{3}}(K_1+K_2+K_3)}|\Psi\rangle = |\Psi\rangle,\\ &\mathrm{e}^{\frac{\pi\mathrm{i}}{2}(L_3+3K_3)}|\Psi\rangle = -|\Psi\rangle. \end{aligned} \tag{47}$$

One can check that the wavefunction (45) satisfies these constraints. Therefore spin $\frac{1}{2}$ is again allowed.

To find the baryon density corresponding to the most interesting case (2) above, let us perturb the rational map (8) to

$$R(z) = \frac{7z^5 + 1 + a}{z^2((1 + a)z^5 - 7)}, \tag{48}$$

which is still D_5-symmetric. Then, to linear order in a, the classical baryon density gets an additional contribution

$$\begin{aligned} \mathcal{B} = 196a|z|^2(1 + |z|^2)^2 \Bigg\{ &\frac{(z^{10} + 11z^5 - 1)(\bar{z}^{10} + \bar{z}^5 - 1) + (z^{10} + z^5 - 1)(\bar{z}^{10} + 11\bar{z}^5 - 1)}{(|z|^{14} + 49|z|^{10} + 49|z|^4 + 1 - 7(|z|^4 - 1)(z^5 + \bar{z}^5))^2} \\ &- 2\frac{(z^{10} + 11z^5 - 1)(\bar{z}^{10} + 11\bar{z}^5 - 1)(2|z|^{14} + 2 - 7(|z|^4 - 1)(z^5 + \bar{z}^5))}{(|z|^{14} + 49|z|^{10} + 49|z|^4 + 1 - 7(|z|^4 - 1)(z^5 + \bar{z}^5))^3} \Bigg\}. \end{aligned} \tag{49}$$

This gives rise to new, non-icosahedrally symmetric terms in the spherical harmonic expansion

$$\mathcal{B} = \sum_{l,m} c_{lm} Y_{lm}(\theta, \phi). \tag{50}$$

The terms which give 90% of the new contribution are $c_{20} \simeq -12a$, $c_{40} \simeq -4a$ and the corrections to c_{60} and c_{65} which are $\Delta c_{60} \simeq 5a$ and $\Delta c_{65} \simeq -4.7a$. Since we are considering small a, the change in the classical baryon density is small. When performing the quantization, the situation changes dramatically. Now we are working with a $J = \frac{3}{2}$ wavefunction. The series for the quantum baryon density will be finite:

$$\rho_\Psi = c_{00}Y_{00} + 0.25c_{20}Y_{20}. \tag{51}$$

Thus, we have a very slightly deformed spherically symmetric density distribution. The density, for $a = 0.1$, is shown in figure 3.

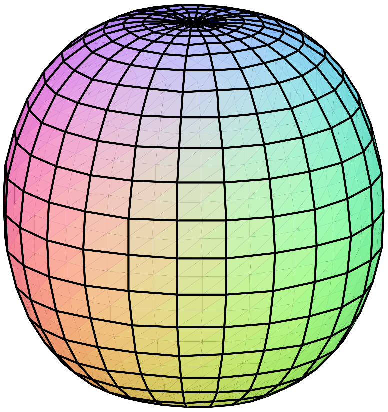

Figure 3. Quantum baryon density of deformed skyrmion with residual D_5 symmetry, in spin $\frac{3}{2}$ state. Effect of deformation a is exaggerated 10 times.

6. Conclusion

In this paper, we have considered a fundamentally new approach to the quantization of the $B = 7$ skyrmion. We have shown that the problem of inconsistency between the experimental ground state and the ground state coming from collective coordinate quantization of the icosahedrally symmetric skyrmion might be overcome by breaking part of the symmetry. We have focused on three different unbroken subgroups of Y_h, and have found the corresponding lowest allowed spin states, and the wavefunctions describing these. For the case where the symmetry is broken to D_5, the lowest allowed spin is $J = \frac{3}{2}$, so this type of deformed skyrmion is the best candidate for modelling the ground states of the ^{7}Li/^{7}Be isospin doublet. Encouragingly, Baskerville [19] found that the lowest frequency, parity-preserving, vibrational mode of the $B = 7$ skyrmion is a squashing and stretching mode preserving D_5 symmetry. This means that for a given amplitude of deformation, the extra potential energy is rather small. Other deformed skyrmions, retaining other symmetries, have both $J = \frac{1}{2}$ and $J = \frac{3}{2}$ spin states. It would be interesting to actually determine the state with the lowest total energy, but that requires a more quantitative investigation of deformation energies and moments of inertia, the topic for some future work. In any case, we now have a promising approach to match the properties of the real Lithium-7 and Beryllium-7 nuclei with the results coming from the previously problematic $B = 7$ sector of the Skyrme model.

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